Lecture 3
The Laplace transform

- definition & examples
- properties & formulas
  - linearity
  - the inverse Laplace transform
  - time scaling
  - exponential scaling
  - time delay
  - derivative
  - integral
  - multiplication by t
  - convolution

Idea

the Laplace transform converts *integral* and *differential* equations into *algebraic* equations

this is like phasors, but

- applies to general signals, not just sinusoids
- handles non-steady-state conditions

allows us to analyze

- LCCODEs
- complicated circuits with sources, Ls, Rs, and Cs
- complicated systems with integrators, differentiators, gains

Complex numbers

complex number in Cartesian form: $z = x + jy$

- $x = \Re z$, the *real part* of z
- $y = \Im z$, the *imaginary part* of z
- $j = \sqrt{-1}$ (engineering notation); $i = \sqrt{-1}$ is polite term in mixed company

complex number in polar form: $z = re^{j\phi}$

- r is the *modulus* or *magnitude* of z
- ϕ is the *angle* or *phase* of z
- $\exp(j\phi) = \cos\phi + j\sin\phi$

complex exponential of $z = x + jy$:

$$e^z = e^{x+jy} = e^x e^{jy} = e^x(\cos y + j\sin y)$$

The Laplace transform

we'll be interested in signals defined for $t \geq 0$

the **Laplace transform** of a signal (function) f is the function $F = \mathcal{L}(f)$ defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

for those $s \in \mathbf{C}$ for which the integral makes sense

- F is a complex-valued function of complex numbers
- s is called the (complex) *frequency variable*, with units sec^{-1}; t is called the *time variable* (in sec); st is unitless
- for now, we assume f contains no impulses at $t = 0$

common notation convention: lower case letter denotes signal; capital letter denotes its Laplace transform, *e.g.*, U denotes $\mathcal{L}(u)$, $V_{\rm in}$ denotes $\mathcal{L}(v_{\rm in})$, etc.

Example

let's find Laplace transform of $f(t) = e^t$:

$$F(s) = \int_0^\infty e^t\, e^{-st}\, dt = \int_0^\infty e^{(1-s)t}\, dt = \frac{1}{1-s} e^{(1-s)t} \Big|_0^\infty = \frac{1}{s-1}$$

provided we can say $e^{(1-s)t} \to 0$ as $t \to \infty$, which is true for $\Re s > 1$:

$$\left| e^{(1-s)t} \right| = \underbrace{\left| e^{-j(\Im s)t} \right|}_{=1} \left| e^{(1-\Re s)t} \right| = e^{(1-\Re s)t}$$

- the *integral* defining F makes sense for all $s \in \mathbf{C}$ with $\Re s > 1$ (the *'region of convergence'* of F)
- but the resulting *formula* for F makes sense for all $s \in \mathbf{C}$ *except* $s = 1$

we'll ignore these (sometimes important) details and just say that

$$\mathcal{L}(e^t) = \frac{1}{s-1}$$

More examples

constant: (or unit step) $f(t) = 1$ (for $t \geq 0$)

$$F(s) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

provided we can say $e^{-st} \to 0$ as $t \to \infty$, which is true for $\Re s > 0$ since

$$\left|e^{-st}\right| = \underbrace{\left|e^{-j(\Im s)t}\right|}_{=1} \left|e^{-(\Re s)t}\right| = e^{-(\Re s)t}$$

- the *integral* defining F makes sense for all s with $\Re s > 0$
- but the resulting *formula* for F makes sense for all s *except* $s = 0$

sinusoid: first express $f(t) = \cos\omega t$ as

$$f(t) = (1/2)e^{j\omega t} + (1/2)e^{-j\omega t}$$

now we can find F as

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}\left((1/2)e^{j\omega t} + (1/2)e^{-j\omega t}\right)\, dt \\ &= (1/2)\int_0^\infty e^{(-s+j\omega)t}\, dt + (1/2)\int_0^\infty e^{(-s-j\omega)t}\, dt \\ &= (1/2)\frac{1}{s-j\omega} + (1/2)\frac{1}{s+j\omega} \\ &= \frac{s}{s^2+\omega^2} \end{aligned}$$

(valid for $\Re s > 0$; final formula OK for $s \neq \pm j\omega$)

powers of t: $f(t) = t^n$ $(n \geq 1)$

we'll integrate by parts, *i.e.*, use

$$\int_a^b u(t)v'(t)\, dt = u(t)v(t)\Big|_a^b - \int_a^b v(t)u'(t)\, dt$$

with $u(t) = t^n$, $v'(t) = e^{-st}$, $a = 0$, $b = \infty$

$$\begin{aligned} F(s) = \int_0^\infty t^n e^{-st}\, dt \quad &= \quad t^n \left(\frac{-e^{-st}}{s}\right)\Bigg|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\, dt \\ &= \quad \frac{n}{s}\,\mathcal{L}(t^{n-1}) \end{aligned}$$

provided $t^n e^{-st} \to 0$ if $t \to \infty$, which is true for $\Re s > 0$

applying the formula recusively, we obtain

$$F(s) = \frac{n!}{s^{n+1}}$$

valid for $\Re s > 0$; final formula OK for all $s \neq 0$

Impulses at $t = 0$

if f contains impulses at $t = 0$ we choose to *include* them in the integral defining F:

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

(you can also choose to not include them, but this changes some formulas we'll see & use)

example: impulse function, $f = \delta$

$$F(s) = \int_{0-}^{\infty} \delta(t)e^{-st}\,dt = e^{-st}\big|_{t=0} = 1$$

similarly for $f = \delta^{(k)}$ we have

$$F(s) = \int_{0-}^{\infty} \delta^{(k)}(t)e^{-st}\,dt = (-1)^k \frac{d^k}{dt^k}e^{-st}\bigg|_{t=0} = s^k e^{-st}\big|_{t=0} = s^k$$

Linearity

the Laplace transform is *linear*: if f and g are any signals, and a is any scalar, we have

$$\mathcal{L}(af) = aF, \qquad \mathcal{L}(f+g) = F + G$$

$i.e.$, homogeneity & superposition hold

example:

$$\begin{aligned} \mathcal{L}\left(3\delta(t) - 2e^t\right) &= 3\mathcal{L}(\delta(t)) - 2\mathcal{L}(e^t) \\ &= 3 - \frac{2}{s-1} \\ &= \frac{3s-5}{s-1} \end{aligned}$$

One-to-one property

the Laplace transform is *one-to-one*: if $\mathcal{L}(f) = \mathcal{L}(g)$ then $f = g$
(well, almost; see below)

- F determines f
- inverse Laplace transform $\mathcal{L}^{-1}$ is well defined

(not easy to show)

example (previous page):

$$\mathcal{L}^{-1}\left(\frac{3s-5}{s-1}\right) = 3\delta(t) - 2e^t$$

in other words, the *only* function f such that

$$F(s) = \frac{3s-5}{s-1}$$

is $f(t) = 3\delta(t) - 2e^t$

what 'almost' means: if f and g differ only at a finite number of points (where there aren't impulses) then $F = G$

examples:

- f defined as
$$f(t) = \begin{cases} 1 & t = 2 \\ 0 & t \neq 2 \end{cases}$$
has $F = 0$

- f defined as
$$f(t) = \begin{cases} 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$
has $F = 1/s$ (same as unit step)

Inverse Laplace transform

in principle we can recover f from F via

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} \, ds$$

where σ is large enough that $F(s)$ is defined for $\Re s \geq \sigma$

surprisingly, this formula isn't really useful!

Time scaling

define signal g by $g(t) = f(at)$, where $a > 0$; then

$$G(s) = (1/a)F(s/a)$$

makes sense: times are scaled by a, frequencies by $1/a$

let's check:

$$G(s) = \int_0^\infty f(at)e^{-st}\,dt = (1/a)\int_0^\infty f(\tau)e^{-(s/a)\tau}\,d\tau = (1/a)F(s/a)$$

where $\tau = at$

example: $\mathcal{L}(e^t) = 1/(s-1)$ so

$$\mathcal{L}(e^{at}) = (1/a)\frac{1}{(s/a)-1} = \frac{1}{s-a}$$

Exponential scaling

let f be a signal and a a scalar, and define $g(t) = e^{at}f(t)$; then

$$G(s) = F(s-a)$$

let's check:

$$G(s) = \int_0^\infty e^{-st}e^{at}f(t)\;dt = \int_0^\infty e^{-(s-a)t}f(t)\;dt = F(s-a)$$

example: $\mathcal{L}(\cos t) = s/(s^2+1)$, and hence

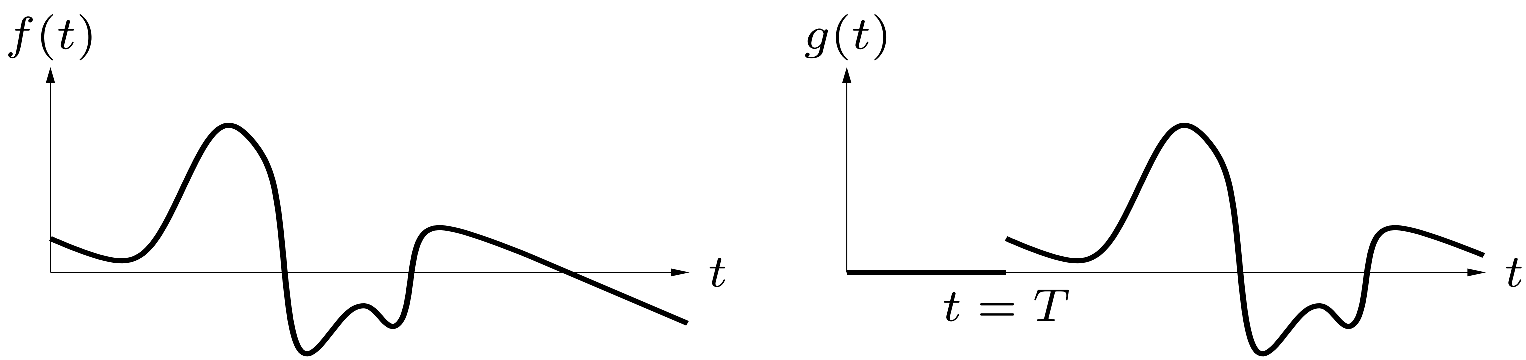
$$\mathcal{L}(e^{-t}\cos t) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$$

Time delay

let f be a signal and $T > 0$; define the signal g as

$$g(t) = \begin{cases} 0 & 0 \leq t < T \\ f(t-T) & t \geq T \end{cases}$$

(g is f, delayed by T seconds & 'zero-padded' up to T)



then we have $G(s) = e^{-sT}F(s)$

derivation:

$$\begin{aligned} G(s) = \int_0^\infty e^{-st} g(t)\, dt &= \int_T^\infty e^{-st} f(t-T)\, dt \\ &= \int_0^\infty e^{-s(\tau+T)} f(\tau)\, d\tau \\ &= e^{-sT}F(s) \end{aligned}$$

example: let's find the Laplace transform of a rectangular pulse signal

$$f(t) = \begin{cases} 1 & \text{if } a \le t \le b \\ 0 & \text{otherwise} \end{cases}$$

where $0 < a < b$

we can write f as $f = f_1 - f_2$ where

$$f_1(t) = \begin{cases} 1 & t \ge a \\ 0 & t < a \end{cases} \qquad f_2(t) = \begin{cases} 1 & t \ge b \\ 0 & t < b \end{cases}$$

i.e., f is a unit step delayed a seconds, minus a unit step delayed b seconds

hence

$$\begin{aligned} F(s) &= \mathcal{L}(f_1) - \mathcal{L}(f_2) \\ &= \frac{e^{-as} - e^{-bs}}{s} \end{aligned}$$

(can check by direct integration)

Derivative

if signal f is continuous at $t = 0$, then

$$\mathcal{L}(f') = sF(s) - f(0)$$

- time-domain differentiation becomes multiplication by frequency variable s (as with phasors)
- *plus* a term that includes initial condition ($i.e.$, $-f(0)$)

higher-order derivatives: applying derivative formula twice yields

$$\begin{aligned}
\mathcal{L}(f'') &= s\mathcal{L}(f') - f'(0) \\
&= s(sF(s) - f(0)) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0)
\end{aligned}$$

similar formulas hold for $\mathcal{L}(f^{(k)})$

examples

- $f(t) = e^t$, so $f'(t) = e^t$ and

$$\mathcal{L}(f) = \mathcal{L}(f') = \frac{1}{s-1}$$

  using the formula, $\mathcal{L}(f') = s(\frac{1}{s-1}) - 1$, which is the same

- $\sin \omega t = -\frac{1}{\omega} \frac{d}{dt} \cos \omega t$, so

$$\mathcal{L}(\sin \omega t) = -\frac{1}{\omega}\left(s\frac{s}{s^2+\omega^2} - 1\right) = \frac{\omega}{s^2+\omega^2}$$

- f is unit ramp, so f' is unit step

$$\mathcal{L}(f') = s\left(\frac{1}{s^2}\right) - 0 = 1/s$$

derivation of derivative formula: start from the defining integral

$$G(s) = \int_0^\infty f'(t) e^{-st} dt$$

integration by parts yields

$$\begin{aligned} G(s) &= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})\, dt \\ &= \lim_{t\to\infty} f(t)e^{-st} - f(0) + sF(s) \end{aligned}$$

for $\Re s$ large enough the limit is zero, and we recover the formula

$$G(s) = sF(s) - f(0)$$

derivative formula for discontinuous functions

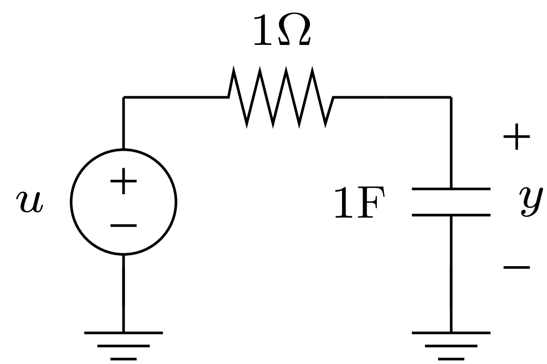
if signal f is discontinuous at $t = 0$, then

$$\mathcal{L}(f') = sF(s) - f(0-)$$

example: f is unit step, so $f'(t) = \delta(t)$

$$\mathcal{L}(f') = s\left(\frac{1}{s}\right) - 0 = 1$$

Example: RC circuit



- capacitor is uncharged at $t = 0$, *i.e.*, $y(0) = 0$
- $u(t)$ is a unit step

from last lecture,

$$y'(t) + y(t) = u(t)$$

take Laplace transform, term by term:

$$sY(s) + Y(s) = 1/s$$

(using $y(0) = 0$ and $U(s) = 1/s$)

solve for $Y(s)$ (just algebra!) to get

$$Y(s) = \frac{1/s}{s+1} = \frac{1}{s(s+1)}$$

to find y, we first express Y as

$$Y(s) = \frac{1}{s} - \frac{1}{s+1}$$

(check!)

therefore we have

$$y(t) = \mathcal{L}^{-1}(1/s) - \mathcal{L}^{-1}(1/(s+1)) = 1 - e^{-t}$$

Laplace transform turned a *differential equation* into an *algebraic equation* (more on this later)

Integral

let g be the running integral of a signal f, $i.e.$,

$$g(t) = \int_0^t f(\tau)\, d\tau$$

then

$$G(s) = \frac{1}{s}F(s)$$

$i.e.$, *time-domain integral* becomes *division by frequency variable* s

example: $f = \delta$, so $F(s) = 1$; g is the unit step function

$$G(s) = 1/s$$

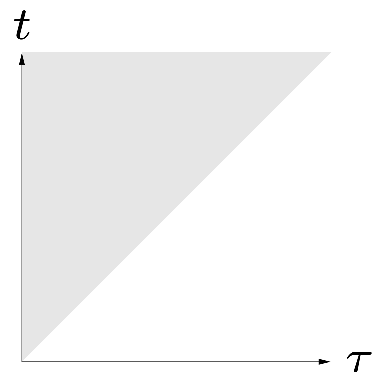
example: f is unit step function, so $F(s) = 1/s$; g is the *unit ramp function* ($g(t) = t$ for $t \geq 0$),

$$G(s) = 1/s^2$$

derivation of integral formula:

$$G(s) = \int_{t=0}^{\infty} \left(\int_{\tau=0}^{t} f(\tau)\, d\tau \right) e^{-st}\, dt = \int_{t=0}^{\infty} \int_{\tau=0}^{t} f(\tau) e^{-st}\, d\tau\, dt$$

here we integrate horizontally first over the triangle $0 \le \tau \le t$



let's switch the order, *i.e.*, integrate vertically first:

$$\begin{aligned} G(s) = \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} f(\tau) e^{-st}\, dt\, d\tau &= \int_{\tau=0}^{\infty} f(\tau) \left(\int_{t=\tau}^{\infty} e^{-st}\, dt \right) d\tau \\ &= \int_{\tau=0}^{\infty} f(\tau)(1/s) e^{-s\tau} d\tau \\ &= F(s)/s \end{aligned}$$

Multiplication by t

let f be a signal and define

$$g(t) = tf(t)$$

then we have

$$G(s) = -F'(s)$$

to verify formula, just differentiate both sides of

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

with respect to s to get

$$F'(s) = \int_0^\infty (-t) e^{-st} f(t)\, dt$$

examples

- $f(t) = e^{-t}$, $g(t) = te^{-t}$

$$\mathcal{L}(te^{-t}) = -\frac{d}{ds}\,\frac{1}{s+1} = \frac{1}{(s+1)^2}$$

- $f(t) = te^{-t}$, $g(t) = t^2e^{-t}$

$$\mathcal{L}(t^2e^{-t}) = -\frac{d}{ds}\,\frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

- in general,

$$\mathcal{L}(t^ke^{-t}) = \frac{(k-1)!}{(s+1)^{k+1}}$$

Convolution

the *convolution* of signals f and g, denoted $h = f * g$, is the signal

$$h(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau$$

- same as $h(t) = \displaystyle\int_0^t f(t-\tau)g(\tau)\, d\tau$; in other words,

$$f * g = g * f$$

- (very great) importance will soon become clear

in terms of Laplace transforms:

$$H(s) = F(s)G(s)$$

Laplace transform turns *convolution* into *multiplication* let's show that $\mathcal{L}(f * g) = F(s)G(s)$:

$$\begin{aligned} H(s) &= \int_{t=0}^{\infty} e^{-st} \left(\int_{\tau=0}^{t} f(\tau)g(t-\tau)\, d\tau \right) dt \\ &= \int_{t=0}^{\infty} \int_{\tau=0}^{t} e^{-st} f(\tau)g(t-\tau)\, d\tau\, dt \end{aligned}$$

where we integrate over the triangle $0 \le \tau \le t$

- change order of integration: $H(s) = \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} e^{-st} f(\tau)g(t-\tau)\, dt\, d\tau$
- change variable t to $\bar{t} = t - \tau$; $d\bar{t} = dt$; region of integration becomes $\tau \ge 0$, $\bar{t} \ge 0$

$$\begin{aligned} H(s) &= \int_{\tau=0}^{\infty} \int_{\bar{t}=0}^{\infty} e^{-s(\bar{t}+\tau)} f(\tau)g(\bar{t})\, d\bar{t}\, d\tau \\ &= \left(\int_{\tau=0}^{\infty} e^{-s\tau} f(\tau)\, d\tau \right) \left(\int_{\bar{t}=0}^{\infty} e^{-s\bar{t}} g(\bar{t})\, d\bar{t} \right) \\ &= F(s)G(s) \end{aligned}$$

examples

- $f = \delta$, $F(s) = 1$, gives
$$H(s) = G(s),$$
which is consistent with
$$\int_0^t \delta(\tau) g(t-\tau) d\tau = g(t)$$

- $f(t) = 1$, $F(s) = e^{-sT}/s$, gives
$$H(s) = G(s)/s$$
which is consistent with
$$h(t) = \int_0^t g(\tau)\, d\tau$$

- more interesting examples later in the course . . .

Finding the Laplace transform

you should *know* the Laplace transforms of some basic signals, $e.g.$,

- unit step ($F(s) = 1/s$), impulse function ($F(s) = 1$)
- exponential: $\mathcal{L}(e^{at}) = 1/(s-a)$
- sinusoids $\mathcal{L}(\cos \omega t) = s/(s^2 + \omega^2)$, $\mathcal{L}(\sin \omega t) = \omega/(s^2 + \omega^2)$

these, combined with a table of Laplace transforms and the properties given above (linearity, scaling, . . .) will get you pretty far

and of course you can always integrate, using the defining formula

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt \quad \ldots$$

Patterns

while the details differ, you can see some interesting symmetric patterns between

- the time domain (*i.e.*, signals), and
- the frequency domain (*i.e.*, their Laplace transforms)

- differentiation in one domain corresponds to multiplication by the variable in the other
- multiplication by an exponential in one domain corresponds to a shift (or delay) in the other

we'll see these patterns (and others) throughout the course