7

The Laplace Transform*

Key words: integral transform, numerical inversion, PDE, ODE

In this chapter, we illustrate the use of the Laplace transform in option pricing. Using the Laplace transform method we can transform a PDE into an ordinary differential equation (ODE) that in general is easier to solve. The solution of the PDE can be then obtained inverting the Laplace transform. Unfortunately when we consider interesting examples, such as pricing Asian options, usually it is difficult to find an analytical expression for the inverse Laplace transform. Then the necessity of the numerical inversion. For this reason, in this chapter we also discuss the problem of the numerical inversion, presenting the Fourier series algorithm that can be easily implemented in MATLAB® or VBA®. The numerical inversion is often disbelieved generically referring to its "intrinsic instability" or for "its inefficiency from a computational point of view". So the aim of this chapter is also to illustrate that the numerical inversion is feasible, is accurate and is not computational intensive. For these reasons, we believe that the Laplace transform instrument will gain greater importance in the Finance field, as already happened in engineering and physics.

In Sect. 7.1 we define the Laplace transform and we give its main properties. In Sect. 7.2, we illustrate the numerical inversion problem. Section 7.3 illustrates a simple application to finance.

7.1 Definition and Properties

In this section we give the basic definition and the properties of the Laplace transform. We say that a function F is of exponential order, if there exist some constants, M and k, for which $|F(\tau)| \leq M e^{k\tau}$ for all $\tau \geq 0$. The Laplace transform $\widehat{F}(\gamma)$ of a

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function $F(\tau)$ is defined by the following integral:

$$\widehat{F}(\gamma) = \mathcal{L}(F(\tau)) = \int_0^{+\infty} \mathrm{e}^{-\gamma\tau} F(\tau)\,\mathrm{d}\tau \tag{7.1}$$

where γ is a complex number and $F(\tau)$ is any function which, for some value of γ, makes the integral finite. The integral (7.1) then exists for a whole interval of values of γ, so that the function $\widehat{F}(\gamma)$ is defined. The integral converges in a right-plane $\mathrm{Re}(\gamma) > \gamma_0$ and diverges for $\mathrm{Re}(\gamma) < \gamma_0$. The number γ_0, which may be $+\infty$ or $-\infty$, is called the *abscissa of convergence*.

Not every function of τ has a Laplace transform, because the defining integral can fail to converge. For example, the functions $1/\tau$, $\exp(\tau^2)$, $\tan(\tau)$ do not possess Laplace transforms. A large class of functions that possess a Laplace transform are of exponential order. Then the Laplace transform of $F(\tau)$ surely exists if the real part of γ is greater than k. In this case, k coincides with the abscissa of convergence γ_0. Also there are certain functions that cannot be Laplace transforms, because they do not satisfy the property $\widehat{F}(+\infty) = 0$, e.g. $\widehat{F}(\gamma) = \gamma$. An important fact is the uniqueness of the representation (7.1), i.e. a function $\widehat{F}(\gamma)$ cannot be the transform of more than one continuous function $F(\tau)$. We have indeed:

Theorem 1 *Let $F(\tau)$ be a continuous function, $0 < \tau < \infty$ and $\widehat{F}(\gamma) \equiv 0$, for $\gamma_0 < \mathrm{Re}(\gamma) < \infty$. Then we have $F(\tau) \equiv 0$.*

In Table 7.1 we give the most important properties of the Laplace transform. In particular, we stress the linearity property

$$\mathcal{L}\big(aF_1(\tau) + bF_2(\tau)\big) = a\mathcal{L}(F_1(\tau)) + b\mathcal{L}(F_2(\tau)),$$

and the Laplace transform of a derivative

$$\mathcal{L}(\partial_\tau F(\tau)) = \gamma\mathcal{L}(F(\tau)) - F(0).$$

In Table 7.2 we give several examples of the Laplace transform $\widehat{F}(\gamma)$ and the corresponding function $F(\tau)$.

If the Laplace transform is known, the original function $F(\tau)$ can be recovered using the inversion formula (Bromwich inversion formula), that can be represented as an integral in the complex plane. We have the following result:

Theorem 2 *If the Laplace transform of $F(\tau)$ exists and has abscissa of convergence with real part γ_0, then for $\tau > 0$*

$$F(\tau) = \mathcal{L}^{-1}(\widehat{F}(\gamma)) = \lim_{R\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}R}^{a+\mathrm{i}R} \widehat{F}(\gamma)\mathrm{e}^{\tau\gamma}\,\mathrm{d}\gamma,$$

where a is another real number such that $a > \gamma_0$ and i is the imaginary unit, $\mathrm{i} = \sqrt{-1}$.

Table 7.1. Basic properties of the Laplace transform

Property	Function	Laplace transform
Definition	$F(\tau)$	$\widehat{F}(\gamma) = \int_0^{+\infty} e^{-\gamma\tau} F(\tau)\, d\tau$
Linearity	$aF_1(\tau) + bF_2(\tau)$	$a\widehat{F}_1(\gamma) + b\widehat{F}_2(\gamma)$
Scale	$aF(a\tau)$	$\widehat{F}(\gamma/a)$
Shift	$e^{a\tau} F(\tau)$	$\widehat{F}(\gamma - a)$
Shift	$\begin{cases} F(\tau - a), & \tau > a \\ 0, & \tau < a \end{cases}$	$e^{-a\gamma}\widehat{F}(\gamma)$
Time derivative	$\frac{\partial F(\tau)}{\partial \tau}$	$\gamma \widehat{F}(\gamma) - F(\tau)\vert_{\tau=0}$
Differentiation	$\frac{\partial^n F(\tau)}{\partial \tau^n}$	$\gamma^n \widehat{F}(\gamma) - \gamma^{n-1} F(0) + \cdots - \gamma^{n-2} F'(0) - \cdots - F^{(n-1)}(0)$
Integral	$\int_0^\tau F(s)\, ds$	$\frac{\widehat{F}(\gamma)}{\gamma}$
Multiplication by polynomials	$\tau^n F(\tau)$	$(-1)^n \widehat{F}^{(n)}(\gamma)$
Convolution	$\int_0^\tau F(s) G(\tau - s)\, ds$	$\widehat{F}(\gamma)\widehat{G}(\gamma)$
Ratio of polynomials	$\sum_{k=1}^n \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k \tau}$	$\sum_{k=1}^n \frac{P(\gamma)}{Q(\gamma)}$
	$P(x)$ polynomial of degree $< n$; $Q(x) = (x - a_1)(x - a_2)\cdots(x - a_n)$ where $a_1 \neq a_1 \neq \cdots \neq a_n$	
Final value	$\lim_{\tau\to\infty} F(\tau)$	$\lim_{\gamma\to 0} \gamma \widehat{F}(\gamma)$
Initial value	$\lim_{\tau\to 0} F(\tau)$	$\lim_{\gamma\to\infty} \gamma \widehat{F}(\gamma)$
Inversion	$\lim_{k\to\infty} \frac{1}{2\pi i} \int_{a-ik}^{a+ik} f(\gamma) e^{\tau\gamma}\, d\gamma$	$\widehat{F}(\gamma)$
	where c is the real part of the rightmost singularity in the image function	

Table 7.2. Some Laplace transforms and their inverses. The function $\delta(t)$ is the delta-Dirac function, the function $J_n(x)$ is the Bessel function of the first kind and of order n, $\mathrm{Erfc}(x)$ is the complementary error function, i.e. $\mathrm{Erfc}(x) = 2N(-\sqrt{2}x)$, where $N(x)$ is the cumulative normal distribution

	$\widehat{F}(\gamma)$	$F(\tau)$
1	1	$\delta(\tau)$
2	$e^{-a\gamma}$	$\delta(\tau - a)$
3	$\frac{1}{\gamma}$	1
4	$\frac{1}{\gamma^2}$	τ
5	$\frac{1}{\gamma^n}, n > 0$	$\frac{\tau^{n-1}}{\Gamma(n)}$
6	$\frac{1}{(\gamma-a)^n}, n > 0$	$\frac{\tau^{n-1} e^{a\tau}}{(n-1)!}$
7	$\frac{1}{\sqrt{\gamma-a}+b}$	$e^{a\tau}\left(\frac{1}{\sqrt{\pi\tau}} - b e^{b^2\tau}\,\mathrm{Erfc}(b\sqrt{\tau})\right)$
8	$\frac{e^{-\vert a\vert\sqrt{\gamma}}}{\sqrt{\gamma}}$	$\frac{e^{-a^2/4\tau}}{\sqrt{\pi\tau}}$
9	$e^{-\vert a\vert\sqrt{\gamma}}$	$\frac{a e^{-a^2/4\tau}}{2\sqrt{\pi\tau^3}}$
10	$\frac{e^{-a\sqrt{\gamma}}}{\sqrt{\gamma}(\sqrt{\gamma}+b)}$	$e^{b(b\tau+a)}\,\mathrm{Erfc}(b\sqrt{\tau} + \frac{a}{2\sqrt{\tau}})$
11	$\frac{e^{-a/\gamma}}{\gamma^{n+1}}$	$(\frac{\tau}{a})^{n/2} J_n(2\sqrt{a\tau})$

The real number a must be selected so that all the singularities of the image function $\widehat{F}(\gamma)$ are to the left of the vertical line $\gamma = \gamma_0$. The integral in the complex plane can be sometimes evaluated analytically using the Cauchy's residue theorem. But this goes beyond an elementary treatment of the Laplace transform and we refer the reader to some textbooks on complex analysis, such as Churchill and Brown (1989). Moreover, this analytical technique often fails and the Bromwich's integral must be integrated numerically.

7.2 Numerical Inversion

Aim of this section is to illustrate how simple and accurate can be the numerical inversion of the Laplace transform. The general opinion that the inversion of the Laplace transform is an ill-conditioned problem[1,2] is due to one of the first tentatives of inversion that reduce the inversion problem to the solution of an ill-conditioned linear system. If we consider a quadrature formula for the integral defining the Laplace transform, we get

$$\widehat{F}(\gamma) = \sum_{i=1}^{n} w_i \mathrm{e}^{-\gamma \tau_i} F(\tau_i). \tag{7.2}$$

Writing this equation for n different values of γ, where γ is supposed to be a *real* number, we are left with an $n \times n$ linear system to be solved wrt the n unknown values $F(\tau_i)$. Unfortunately, the solution of this linear system can change abruptly given little changes in $\widehat{F}(\gamma)$. The ill-conditioning of the above inversion method is common to all numerical routines that try the inversion computing the Laplace transform only for real values of the parameter γ. The exponential kernel that appears in the definition of the Laplace transform smooths out too much the original function. Therefore, to recover $F(\tau)$ given values of the Laplace transform on the real axis can be very difficult. This problem occurs when $\widehat{F}(\gamma)$ is the result of some physical experiment, so that it can be affected by measurement errors. Instead, this problem does not arise when the Laplace transform is known in closed form as a complex function. In this case instead of discretizing the integral defining the forward Laplace transform, we

[1] The concept of well-posedness was introduced by Hadamard and, simply stated, it means that a well-posed problem should have a solution, that this solution should be unique and that it should depend continuously on the problem's data. The first two requirements are minimal requirements for a reasonable problem, and the last ensures that perturbations, such errors in measurement, should not unduly affect the solution.

[2] "The inversion of the Laplace transform is well known to be an ill-conditioned problem. Numerical inversion is an unstable process and the difficulties often show up as being highly sensitive to round-off errors", Kwok and Barthez (1989). "The standard inversion formula is a contour integral, not a calculable expression .... These methods provide convergent sequences rather than formal algorithms; they are difficult to implement (many involve solving large, ill-conditioned systems of linear equations or analytically obtaining high-order derivatives of the transform) and none includes explicit, numerically computable bounds on error and computational effort", Platzman, Ammons and Bartholdi (1988).

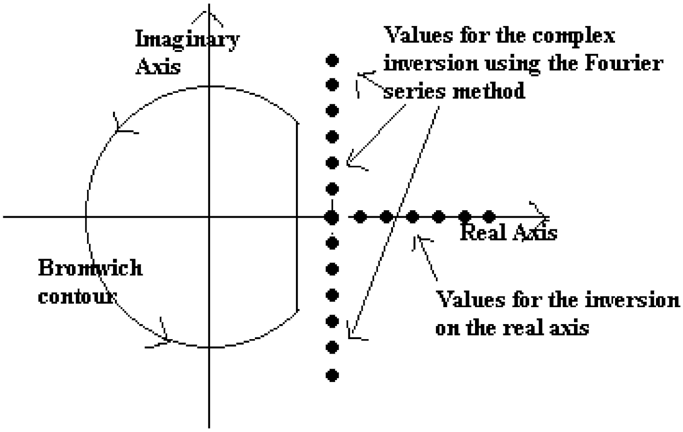

Fig. 7.1. Sample points for the inversion with the Fourier series method using the Bromwich contour and sample points for the inversion on the real axis using the definition of Laplace transform.

can operate the inversion using directly the Bromwich contour integral, and then using values of the transform in the complex plane. The different approach of inverting the Laplace transform on the real axis or on the complex plane is illustrated in Fig. 7.1. This section describe a very effective Laplace inversion algorithm that involves complex calculations.[3]

Letting the contour be any vertical line $\gamma = a$ such that $\widehat{F}(\gamma)$ has no singularities on or to the right of it, the original function $F(\tau)$ is given by the inversion formula:

$$F(\tau) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{e}^{\gamma\tau} \widehat{F}(\gamma)\, \mathrm{d}\gamma, \quad \tau > 0. \tag{7.3}$$

Alternatively, setting $a+\mathrm{i}u = \gamma$ and using the identity from complex variable theory, $\mathrm{e}^{\gamma} = \mathrm{e}^{a}(\cos(u) + \mathrm{i}\sin(u))$, $\mathrm{Re}(\widehat{F}(a+\mathrm{i}u)) = \mathrm{Re}(\widehat{F}(a-\mathrm{i}u))$, $\mathrm{Im}(\widehat{F}(a+\mathrm{i}u)) = -\mathrm{Im}(\widehat{F}(a-\mathrm{i}u))$, $\sin(u\tau) = -\sin(-u\tau)$, $\cos(u\tau) = \cos(-u\tau)$, and from the fact that the integral in (7.3) is 0 for $\tau < 0$, we get

$$F(\tau) = \frac{2\mathrm{e}^{a\tau}}{\pi} \int_{0}^{+\infty} \mathrm{Re}\big(\widehat{F}(a+\mathrm{i}u)\big) \cos(u\tau)\, \mathrm{d}u \tag{7.4}$$

and

$$F(\tau) = -\frac{2\mathrm{e}^{a\tau}}{\pi} \int_{0}^{+\infty} \mathrm{Im}\big(\widehat{F}(a+\mathrm{i}u)\big) \sin(u\tau)\, \mathrm{d}u.$$

$F(\tau)$ can be calculated from (7.4) by performing a numerical integration (quadrature). Since there are many numerical integration algorithms, the remaining goal is

[3] Certain computer languages such as Matlab®, Mathematica, Fortran and C++ have automatic provision for doing complex calculations. In VBA® or C we need instead to define a new type of variable and to say how operations between complex numbers must be performed.

to exploit the special structure of the integrand in (7.4) in order to calculate the integral accurately and efficiently.

The algorithm we describe is named Fourier series method and has received great attention recently in finance, for the simplicity of implementation and the accuracy in the numerical results. The underlying idea of the method is to discretize (7.4) using the trapezoidal rule. Then the inversion is given as a sum of infinite terms. The convergence of the series is accelerated using the Euler algorithm. This algorithm allows one to compute a series with great accuracy using a limited number of terms (in several examples founded in the literature no more than 30).

7.3 The Fourier Series Method

The Fourier series algorithm has been originally proposed by Dubner and Abate (1968) and then improved by Abate and Whitt (1992b). It is essentially a trapezoidal rule approximation to (7.4). An essential feature of this method is that an expression for the error in the computed inverse transform is available. Therefore, one can control the maximum error in the inversion technique. Since the trapezoidal rule is a quite simple integration procedure, its use can appear surprising. It turns out to be surprisingly effective in this context with periodic and oscillating integrands, because the errors tend to cancel. In particular, it turns out to be better than familiar alternatives such as Simpson's rule or Gaussian quadrature for inversion integrals.

If we apply the trapezoidal rule with step size Δ to the expression in (7.4), we get

$$F(\tau) \simeq F_{\Delta}^{\mathrm{DA}}(\tau) = \frac{\Delta e^{a\tau}}{\pi} \operatorname{Re}(\widehat{F}(a)) + \frac{2\Delta e^{a\tau}}{\pi} \sum_{k=1}^{\infty} \operatorname{Re}\big(\widehat{F}(a + \mathrm{i}k\Delta)\big) \cos(k\Delta\tau).$$

If we set $\Delta = \pi/(2\tau)$ and $a = A/(2\tau)$, we can eliminate the cosine terms and we obtain an alternating series

$$F_{\Delta}^{\mathrm{DA}}(\tau) = \frac{e^{A/2}}{2\tau} \operatorname{Re}\left(\widehat{F}\left(\frac{A}{2\tau}\right)\right) + \frac{e^{A/2}}{\tau} \sum_{k=1}^{\infty} (-1)^k \operatorname{Re}\left(\widehat{F}\left(\frac{A + 2k\pi\mathrm{i}}{2\tau}\right)\right). \tag{7.5}$$

The choice of A has to be made in such a way that a falls at the left of the real part of all the singularities of the function $\widehat{F}(\gamma)$ ($a = 0$ suffices when F is a bounded continuous probability density). Assuming that $|F(\tau)| < M$, Abate and Whitt (1992b) show that the discretization error can be bounded by

$$\left|F(\tau) - F_{\Delta}^{\mathrm{DA}}(\tau)\right| < M \frac{e^{-A}}{1 - e^{-A}} \simeq M e^{-A}, \tag{7.6}$$

so that we should set A large in order to make the error small. In order to obtain a discretization error less than $10^{-\delta}$, we can set $A = \delta \ln 10$. However, increasing A can make the inversion (7.5) harder, due to roundoff errors. Thus A should not be chosen too large. In practice, Abate and Whitt (1992b) suggest to set A equal to 18.4.

The remaining problem consists in computing the infinite sum in (7.5). If the term $\mathrm{Re}(\widehat{F}((A+2k\pi\mathrm{i})/(2\tau)))$ has a constant sign for all k, it can be convenient to consider an accelerating algorithm for alternating series. Abate and Whitt (1992b) propose the use of the Euler algorithm. This algorithm consists in summing explicitly the first n terms of the series and then in taking a weighted average of additional m terms. In practice, the Euler algorithm estimates the series using $E(\tau; n, m)$, where

$$F_{\Delta}^{\mathrm{DA}}(\tau) \approx E(\tau; n, m) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(\tau), \tag{7.7}$$

and where $s_n(\tau)$ is the nth partial sum:

$$s_n(\tau) = \frac{\mathrm{e}^{A/2}}{2\tau} \mathrm{Re}\left(\widehat{F}\left(\frac{A}{2\tau}\right)\right) + \frac{\mathrm{e}^{A/2}}{\tau} \sum_{k=1}^{n} (-1)^k \mathrm{Re}\left(\widehat{F}\left(\frac{A+2k\pi\mathrm{i}}{2\tau}\right)\right). \tag{7.8}$$

As pointed out in Abate and Whitt (1992b, p. 46), in order for Euler summation to be effective, $a_k = \mathrm{Re}(\widehat{F}((A+2k\pi\mathrm{i})/(2\tau)))$ must have three properties for sufficiently large k: (a) to be of constant sign, (b) to be monotone, (c) the higher-order differences $(-1)^m \Delta^m a_{n+k}$ are monotone. On a practical side, these properties are not checked, so that the algorithm is used in a heuristic way. Usually, $E(\tau; n, m)$ approximates the true sum with an error of the order of 10^{-13} or less with the choice $n = 38$ and $m = 11$, i.e. using just 50 terms. The direct computation of the series can require more than 10,000 terms. The Abate–Whitt algorithm gives excellent results for functions that are sufficiently smooth (say, twice continuously differentiable). However, the inversion algorithm performs less satisfactorily for points at which the function $f(t)$ or its derivative is not differentiable.

Example Let us test the algorithm with the series $\sum_{k=1}^{+\infty} (-1)^k/k$, that converges to $-\ln 2 = -0.6931471805599453$. Computing the sum using 100,000 terms, we get -0.6931421805849445, i.e. a five digits accuracy. Using the Euler algorithm with $n = 19$ and $n + m = 30$, we get -0.693147180559311, i.e. a ten digits accuracy! This is illustrated in Fig. 7.2.

The procedure for the numerical inversion is then resumed in Table 7.3.

7.4 Applications to Quantitative Finance

In this section we illustrate how the Laplace transform method can be useful in solving linear parabolic equations. We consider two examples: (a) pricing a call option in the standard Black–Scholes model; (b) pricing an Asian option in the square-root model.

7.4.1 Example

For this, let us consider the Black–Scholes PDE satisfied by the price $F(\tau, X)$ of a derivative contract having time to maturity $T - t$

Euler Algorithm for the series Σ(1-)^k/k

m 10

n	$(-1)^k/k$	s_n
1	-1	-1
2	0.5	-0.5
3	-0.333333	-0.833333
4	0.25	-0.583333
5	-0.2	-0.783333
6	0.166667	-0.616667
7	-0.142857	-0.759524
8	0.125	-0.634524
9	-0.111111	-0.745635
10	0.1	-0.645635
11	-0.090909	-0.736544
12	0.083333	-0.653211
13	-0.076923	-0.730134
14	0.071429	-0.658705
15	-0.066667	-0.725372
16	0.0625	-0.662872
17	-0.058824	-0.721695
18	0.055556	-0.66614
19	-0.052632	-0.718771
20	0.05	-0.668771
21	-0.047619	-0.71639
22	0.045455	-0.670936
23	-0.043478	-0.714414
24	0.041667	-0.672747
25	-0.04	-0.712747
26	0.038462	-0.674286
27	-0.037037	-0.711323
28	0.035714	-0.675609
29	-0.034483	-0.710091
30	0.033333	-0.676758

Extra terms	Bin. Coeff.	$\binom{m}{k}2^{-m}s_{n+k}(\tau)$
0	1	-0.000653097
1	10	-0.006996
2	45	-0.029484488
3	120	-0.08372041
4	210	-0.137965796
5	252	-0.175402705
6	210	-0.138281301
7	120	-0.083358164
8	45	-0.029689836
9	10	-0.006934487
10	1	-0.000660897

Euler Sum	-0.69314718056

Fig. 7.2. Euler algorithm for computing $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$.

$$\partial_t F + rx\partial_x F + \frac{1}{2}\sigma^2 x^2 \partial_{xx} F = rF, \qquad (7.9)$$
$$F(T, x) = \phi(x),$$

and appropriate boundary conditions. Let us define

$$\tau = \frac{\sigma^2}{2}(T - t), \qquad z = \ln x,$$

and let us introduce the new function

$$F(t, x) = f(\tau, z).$$

Table 7.3. Pseudo-code for implementing the numerical inversion of the Laplace transform

Define the Laplace Transform $\widehat{F}(\frac{A}{2\tau})$
Assign A, n, m
Compute s_j in (7.8), $j = 1, m + m$.
Using $s_n, \dots, s_{n+m}$ compute $E(\tau; n, m)$

Then f solves the PDE

$$-\partial_\tau f(\tau, z) + \left(\frac{r}{\sigma^2/2} - 1\right)\partial_z f(\tau, z) + \partial_{zz} f(\tau, z) - \frac{r}{\sigma^2/2} f(\tau, z) = 0, \qquad (7.10)$$

with initial condition $f(0, z) = F(T, e^z)$. In the following, we consider as payoff function

$$f(0, z) = F\big(T, e^z\big) = \big(e^z - e^k\big)_+,$$

i.e. a plain vanilla option (and therefore $f(\tau, z) \to e^z - e^k$ as $z \to +\infty$ and $f(\tau, z) \to 0$ as $z \to -\infty$). If we Laplace transform the above partial differential equation with constant coefficients, the result will be an algebraic equation in the transform of the unknown variable. Indeed, from the properties illustrated in Table 7.1, we have

$$\begin{aligned}
\mathcal{L}(f(\tau, z)) &= \int_0^\infty e^{-\gamma\tau} f(\tau, z)\,d\tau = \hat{f}(\gamma, z),\\
\mathcal{L}(\partial_\tau f(\tau, z)) &= \int_0^\infty e^{-\gamma\tau} \partial_\tau f(\tau, z)\,d\tau = \gamma \hat{f}(\gamma, z) - f(0, z),\\
\mathcal{L}(\partial_z f(\tau, z)) &= \int_0^\infty e^{-\gamma\tau} \partial_z f(\tau, z)\,d\tau = \partial_z \hat{f}(\gamma, z),\\
\mathcal{L}(\partial_{zz} f(\tau, z)) &= \int_0^\infty e^{-\gamma\tau} \partial_{zz} f(\tau, z)\,d\tau = \partial_{zz} \hat{f}(\gamma, z).
\end{aligned}$$

Therefore, we have the means of turning the PDE (7.9), for the linearity of the Laplace transform, into the second-order ordinary differential equation (ODE):

$$\begin{aligned}
-\big(\gamma \hat{f}(\gamma, z) - \big(e^z - e^k\big)_+\big) + \left(\frac{r}{\sigma^2/2} - 1\right)\partial_z \hat{f}(\gamma, z)\\
+ \partial_{zz} \hat{f}(\gamma, z) - \frac{r}{\sigma^2/2} \hat{f}(\gamma, z) = 0.
\end{aligned}$$

Then setting $m = r/(\sigma^2/2)$ we get

$$\partial_{zz} \hat{f}(\gamma, z) + (m-1)\partial_z \hat{f}(\gamma, z) - (m+\gamma)\hat{f}(\gamma, z) + \big(e^z - e^k\big)_+ = 0 \qquad (7.11)$$

with boundary conditions given by the Laplace transform of the boundary conditions of the original PDE:

Table 7.4. Laplace transform and PDE

Original space	$\left\{\begin{array}{c}\text{PDE}\\ +\text{IC}\\ +\text{BC's}\end{array}\right\}$		Solution	
			$\nearrow\nwarrow$	
		analytical		numerical
	$\downarrow$		$\nwarrow\nearrow$	
	$\mathcal{L}$-transform		$\mathcal{L}^{-1}$-transform	
	$\downarrow$		$\uparrow$	
Image space	$\left\{\begin{array}{c}\text{ODE}\\ +\text{BC's}\end{array}\right\}$	$\rightarrow$	Solution	

$$\hat{f}(\gamma, z) \to \mathcal{L}\big(\mathrm{e}^z - \mathrm{e}^{-m\tau}\mathrm{e}^k\big) = \frac{\mathrm{e}^z}{\gamma} - \frac{\mathrm{e}^k}{\gamma + m} \quad \text{as } z \to +\infty, \tag{7.12}$$

$$\hat{f}(\gamma, z) \to \mathcal{L}(0) = 0 \quad \text{as } z \to -\infty. \tag{7.13}$$

The initial condition of the PDE has been included in the ODE, where now there is the appearance of the term $(\mathrm{e}^z - \mathrm{e}^k)_+$. Therefore, instead of solving the PDE (7.10) we are left with the second-order differential equation in (7.11), that actually is simpler to solve. Then, the problem will be to recover the solution of the PDE from the solution of the ODE, i.e. to find the inverse Laplace transform. The procedure is illustrated in Table 7.4.

In order to solve (7.11), let us define

$$\hat{f}(\gamma, z) = \exp(\alpha z)\hat{g}(\gamma, z),$$

where $\alpha = (1 - m)/2$. Then $\hat{g}(\gamma, z)$ solves

$$\partial_{zz}\hat{g}(\gamma, z) - (b + \gamma)\hat{g} + \mathrm{e}^{-\alpha z}\big(\mathrm{e}^z - \mathrm{e}^k\big)_+ = 0,$$

with $b = \alpha^2 + m = (m - 1)^2/4 + m$. We can solve this ODE separately in the two regions $z > k$ and $z \le k$ to get

$$\hat{g}(\gamma, z) = \begin{cases} \frac{\mathrm{e}^{-(\alpha-1)z}}{\gamma} - \frac{\mathrm{e}^{-\alpha z + k}}{\gamma + m} + h_1(\gamma, z)A_1 + h_2(\gamma, z)A_2, & z > k, \\ h_1(\gamma, z)B_1 + h_2(\gamma, z)B_2, & z \le k, \end{cases}$$

where

$$h_1(\gamma, z) = \mathrm{e}^{-\sqrt{b+\gamma}\,z}, \qquad h_2(\gamma, z) = \mathrm{e}^{+\sqrt{b+\gamma}\,z}.$$

Here A_1, A_2, B_1 and B_2 are constants to be determined according to the boundary conditions (7.12) and (7.13) and requiring that $\hat{f}(\gamma, z)$ is continuous and differentiable at $z = k$ (*smooth pasting conditions*). We observe that the singularities of $\hat{g}(\gamma, z)$ are 0, $-m$ and $-b$. Therefore the abscissa of convergence of $\hat{g}(\gamma, z)$ is given by

$$\gamma_0 = \max(0, -m, -b).$$

Given that when $\gamma > \gamma_0$, $\lim_{z\to+\infty} \mathrm{e}^{\alpha z} h_1(\gamma, z) = 0$ and $\lim_{z\to+\infty} \mathrm{e}^{\alpha z} h_2(\gamma, z) = \infty$, we must set $A_2 = 0$. Similarly, when $z < k$ we need to set $B_1 = 0$. We are therefore left with

$$\hat{g}(\gamma, z) = \begin{cases} \frac{e^{-(\alpha-1)z}}{\gamma} - \frac{e^{-\alpha z+k}}{\gamma+m} + h_1(\gamma, z)A_1, & z > k, \\ h_2(\gamma, z)B_2, & z \le k, \end{cases}$$

and now we determine A_1 and B_2 with the additional conditions

$$\lim_{z\to k^+} \hat{f}(\gamma, z) = \lim_{z\to k^-} \hat{f}(\gamma, z),$$
$$\lim_{z\to k^+} \partial_z \hat{f}(\gamma, z) = \lim_{z\to k^-} \partial_z \hat{f}(\gamma, z).$$

With some tedious algebra, we get

$$A_1(\gamma) = \frac{e^{(1-a+\sqrt{b+\gamma})k}(\gamma - (a - 1 + \sqrt{b+\gamma})m)}{2\gamma\sqrt{b+\gamma}(\gamma+m)},$$
$$B_2(\gamma) = \frac{e^{(1-a-\sqrt{b+\gamma})k}(\gamma - (a - 1 - \sqrt{b+\gamma})m)}{2\gamma\sqrt{b+\gamma}(\gamma+m)},$$

and finally we obtain the following expression for the function $\hat{f}(\gamma, z)$

$$\hat{f}(\gamma, z) = e^{az}\Bigg[\left(\frac{e^{-(\alpha-1)z}}{\gamma} - \frac{e^{-\alpha z+k}}{\gamma+m}\right)1_{(z>k)} + \frac{e^{-\sqrt{b+\gamma}|z-k|+(1-a)k}(\gamma - (a - 1 + \sqrt{b+\gamma}\,\mathrm{sgn}(z-k))m)}{2\gamma\sqrt{b+\gamma}(\gamma+m)}\Bigg], \tag{7.14}$$

where $\mathrm{sgn}(z) = 1_{(z\ge 0)} - 1_{(z<0)}$.

We can also easily obtain the Laplace transform of the Delta and the Gamma of the option differentiating with respect to $x = e^z$ the Laplace transform.

Numerical Inversion

The numerical inversion has been implemented in MATLAB® and in VBA®. In MATLAB®, we have built the following functions

```
function [lt] = ltbs(spot, strike, sg, rf, gamma)
function [euler] = AWBS(spot, strike, expiry, sg, rf,
aa, terms, extraterms)
```

The function `ltbsm` returns the Laplace transform in (7.14), taking as inputs the spot price (`spot`), the strike (`strike`), the volatility (`sg`), the risk-free rate (`rf`) and the Laplace parameter γ (`gamma`). The function `AWBS` performs the numerical inversion (Fourier series with Euler summation) returning the Black–Scholes price. The parameter `aa` is the constant A that determines the discretization error in (7.6), `terms` is the number of terms n we use to estimate s_n, and `extraterms` is the additional number of terms m needed to perform the Euler summation. Similar functions have been constructed in VBA® for Excel. Here below, we give the Matlab® code.

```
   function [optprice] = AWBS(spot, strike, expiry, sg,
rf, aa, terms, extraterms)
   tau = expiry * sg * sg / 2;
   sum = 0;
   %%compute the LT at gamma = aa / (2 * tau)
   lt = ltbs(spot, strike, sg, rf, aa / (2 * tau));
   sum = lt* exp(aa / 2) / (2 * tau);
   %apply the Euler algorithm
   k = [1:terms + extraterms];
   arg = aa / (2 * tau)+i*pi.*k / tau;
   term = ((-1) .^k) .* ltbs(spot, strike, sg, rf, arg)
* exp(aa / 2) / tau;
   csum = sum+cumsum(term);
   sumr = real(csum(terms:terms+extraterms));
   j=[0:extraterms];
   bincoeff = gamma(extraterms+1)./(gamma(j+1).
* gamma(extraterms-j+1));
   %extrapolated result
   optprice = (bincoeff*sumr')*(2) ^(-extraterms);

   function [lt] = ltbs(spot, strike, sg, rf, gamma)
   m = 2 * rf / (sg * sg); a = (1 - m) / 2; b = a ^2 + m;
   z = log(spot); k = log(strike);
   %%%FORMULA 14: NUMERATOR
   term0 = (b+gamma).^0.5;
   %'the numerator
   if spot >strike
    term1 = term0;
   else
    term1 = -term0;
   end
   term1 = a - 1+term1;
   term1 = m*term1;
   num = gamma-term1;
   %'the denominator
   den = 2.*gamma .* term0.*(m+gamma);
   %'the exponential term
   term2 = exp(k*(1-a)-term0*abs(z-k));
   result = term2.*num./ den;
   if spot > strike
   %'exp(-(a-1)*z)/gamma
    cterm1 = exp(-(a - 1) * z)./gamma;
   %'exp(-a*z+k)/(gamma+m)
    cterm2 = exp(-a * z + k)./(gamma +m);
   %'A1*h1
    result = cterm1-cterm2+result;
   end
   lt = exp(a * z).*result;
```

Table 7.5. Pricing of a call option: analytical Black–Scholes (3rd column) and numerical inversion of the Laplace transform (4th and 5th columns). Parameters: strike = 100, $r = 0.05$, $\sigma = 0.2$

Expiry	Spot	BS	$A = 18.4, n = 15, m = 10$	$A = 18.4, n = 50, m = 10$
0.001	90	0.00000	0.00000	0.00000
0.001	100	0.254814	0.254814	0.254814
0.001	110	10.00500	10.00500	10.00500
0.5	90	2.349428	2.349428	2.349428
0.5	100	6.888729	6.888729	6.888729
0.5	110	14.075384	14.075384	14.075384
1	90	5.091222	5.091222	5.091222
1	100	10.450584	10.450584	10.450584
1	110	17.662954	17.662954	17.662954
5	90	21.667727	21.667727	21.667727
5	100	29.13862	29.13862	29.13862
5	110	37.269127	37.269128	37.269128
20	90	57.235426	57.235426	57.235427
20	100	66.575748	66.575748	66.575749
20	110	76.048090	76.048090	76.048091
		m.s.e.	0.00000147	0.00000203

In Table 7.5 we report the exact Black–Scholes price and the one obtained by numerical inversion. The numbers in Table 7.5 can be obtained running the MATLAB® module `main`.

7.4.2 Example

As a second example, we consider the use of the Laplace transform with respect to the strike and not with respect to the time to maturity. This different approach is possible when the moment generating function (m.g.f.) of the underlying variable is known in closed form. The m.g.f. of a random variable Z is defined as $\mathbb{E}_0[e^{-\gamma Z}]$. In particular, if Z is a non-negative r.v. and admits density $f_Z(z)$, we have

$$\mathbb{E}_0\left[e^{-\gamma Z}\right] = \int_0^{+\infty} e^{-\gamma z} f_Z(z)\, dz,$$

and hence the interpretation of the m.g.f. as Laplace transform of the density function. Notice that the existence of the m.g.f. is not always guaranteed because it is required that the m.g.f. is defined in a complete neighborhood of the origin. For example, this is not the case when Z is lognormal.

If the m.g.f. of the random variable Z is known, we can also obtain the Laplace transform of a call option written on $Z(t)$. Let us consider a contingent claim with payoff given by $\alpha(Z(t) - Y)_+$, where α and Y are constants. By no-arbitrage arguments, the option price is:

$$C\big(Z(0),t,Y\big) = \alpha e^{-rt}\int_0^{+\infty}(z-Y)^+ f_Z(z)\,dz$$
$$= \alpha e^{-rt}\int_Y^{+\infty}(z-Y) f_Z(z)\,dz, \tag{7.15}$$

where $f_Z(z)$ is the risk-neutral density of $Z(t)$. Let us define the Laplace transform wrt Z of the above price

$$c\big(Z(0),t;\gamma\big) = \mathcal{L}\big[C\big(Z(0),t,Y\big)\big] = \int_0^{+\infty} e^{-\gamma Y} C\big(Z(0),t,Y\big)\,dY.$$

Replacing (7.15) in this formula and using a change of integration, we get

$$\begin{aligned}
c\big(Z(0),t;\gamma\big) &= \alpha e^{-rt}\int_0^{+\infty} e^{-\gamma Y}\int_x^{+\infty}(z-Y) f_Z(z)\,dz\,dY\\
&= \alpha e^{-rt}\int_0^{+\infty}\left(\int_0^z e^{-\gamma Y}(z-Y)\,dY\right) f_Z(z)\,dz\\
&= \alpha e^{-rt}\int_0^{+\infty}\left(\int_0^z \big(z e^{-\gamma Y} - Y e^{-\gamma Y}\big)\,dY\right) f_Z(z)\,dz\\
&= \alpha e^{-rt}\int_0^{+\infty}\frac{e^{-\gamma z}+\gamma z-1}{\gamma^2} f_Z(z)\,dz\\
&= \alpha e^{-rt}\left(\frac{\mathbb{E}_0[e^{-\gamma Z(t)}]}{\gamma^2}+\frac{\mathbb{E}_0[Z(t)]}{\gamma}-\frac{1}{\gamma^2}\right).
\end{aligned}$$

Using the fact that the Laplace inverse of $1/\gamma$ is 1 and the Laplace inverse of $1/\gamma^2$ is Y, we can write the option price as follows

$$C\big(Z(0),t,Y\big) = \alpha e^{-rt}\left(\mathcal{L}^{-1}\left(\frac{\mathbb{E}_0[e^{-\gamma Z_t}]}{\gamma^2}\right)+\mathbb{E}_0[Z_t]-Y\right), \tag{7.16}$$

and the pricing problem is reduced to the numerical inversion of $\mathbb{E}_0[e^{-\gamma Z_t}]/\gamma^2$.

As a concrete example, let us consider the square root process

$$dX(t) = rX(t)\,dt + \sigma\sqrt{X(t)}\,dW(t),$$

and our aim is to price a fixed strike Asian call option, having payoff

$$\frac{1}{t}\left(\int_0^t X(u)\,du - Kt\right)_+.$$

In order to obtain the price of the Asian option, we compute the moment generating function of $\int_0^t X(u)\,du$:

$$v\big(X(0),t;\gamma\big) = \mathbb{E}_0\big[e^{-\gamma\int_0^t X(u)\,du}\big]. \tag{7.17}$$

By the Feynman–Kac theorem, $v(X(0), t; \gamma)$ is the solution of the PDE:

$$-\partial_t v + rx\,\partial_x v + \frac{1}{2}\sigma^2 x\,\partial_{xx} v = \gamma x v$$

with initial condition

$$v\big(X(0), 0; \gamma\big) = 1.$$

To solve this PDE, we exploit the linearity of the drift and variance coefficients and, following Ingersoll (1986), pp. 397–398, we consider a solution of the type:

$$v(X, t;\ \gamma) = \mathrm{e}^{-A(t;\ \gamma)X - B(t;\gamma)}.$$

Replacing this function in the PDE, it is then easy to show that $B(t; \gamma) = 0$ and

$$A(t; \gamma) = \frac{2\gamma(\exp(t\lambda) - 1)}{\lambda + r + (\lambda - r)\exp(t\lambda)}, \tag{7.18}$$

where $\lambda = \sqrt{r^2 + 2\gamma\sigma^2}$. Therefore, using (7.16), we can write the price of the Asian option as

$$\alpha \mathrm{e}^{-rt}\left(\mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-A(t;\gamma)X - B(t;\gamma)}}{\gamma^2}\right) + \mathbb{E}_0\left[\int_0^t X(u)\,\mathrm{d}u\right] - X\right),$$

where $\mathcal{L}^{-1}$ is the Laplace inverse. In particular, we have:

$$\begin{aligned}
\mathbb{E}_0\left[\int_0^t X(u)\,\mathrm{d}u\right] &= \int_0^t \mathbb{E}_0[X(u)]\,\mathrm{d}u \\
&= \int_0^t X(0)\mathrm{e}^{ru}\,\mathrm{d}u \\
&= X(0)\frac{\mathrm{e}^{rt} - 1}{r}.
\end{aligned}$$

Numerical inversion

Table 7.6 provides some numerical example. In the numerical inversion of the Laplace transform we have used $A = 18.4$, and the Euler algorithm has been applied using a total of $20 + 10$ terms.

Table 7.6. Prices of an Asian option in the square-root model

K	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.5$
0.9	0.137345	0.15384	0.18691
0.95	0.09294	0.12001	0.15821
1	0.05258	0.09075	0.13253
1.05	0.02268	0.06640	0.10987
1.1	0.00687	0.04696	0.09016

These examples have been obtained writing the Matlab® functions

```
AWSR(spot, strike, expiry, sg, rf, aa, terms,
extraterms)
ltsr(spot, expiry, sg, rf, gamma)
```

The function AWSR performs the numerical inversion of the Laplace transform according to the Abate–Whitt algorithm. The function ltsr returns the quantity $v(X, t; \gamma)/\gamma^2$. The complete code is given here below.

```
function [optprice]=AWSR(spot, strike, expiry, sg, rf,
aa, terms, extraterms)
X = strike*expiry;
sum = 0;
%%compute the LT at gamma = aa / (2 * strike)
lt = ltsr(spot , expiry , sg , rf , aa / (2 * X) );
sum = lt* exp(aa / 2) / (2 * strike);
%apply the Euler algorithm
k = [1:terms + extraterms];
arg = aa / (2 * X)+i*pi.*k / X;
term = ((-1) .^k) .* ltsr(spot, expiry, sg, rf, arg)
* exp(aa / 2) / X;
csum = sum+cumsum(term);
sumr = real(csum(terms:terms+extraterms));
j = [0:extraterms];
bincoeff = gamma(extraterms+1)./(gamma(j+1).
* gamma(extraterms-j+1));
%extrapolated result
euler = (bincoeff*sumr')*(2) ^(-extraterms);
%apply the final formula
optprice = exp(-rf*expiry)*(euler+spot*(exp(rf*expiry)
-1)/rf - X)/expiry;

function [lt] = ltsr(spot, expiry, sg, rf, gamma)
lambda = sqrt(rf^2+2*gamma*sg*sg);
numerator = 2*gamma.*(exp(expiry.*lambda)-1);
denominator = lambda+rf+(lambda-rf).*exp(expiry.
*lambda);
lt = exp(-spot*numerator./denominator)./gamma.^2;
```

7.5 Comments

A good introduction to the Laplace transform topic can be found in Dyke (1999), whilst a classical but more advanced treatment is Doetsch (1970). Extensive tables for analytical inversion of the Laplace transform are available: see for example Abramowitz and Stegun (1965). Davies and Martin (1970) provide a review and a comparison of some numerical inversion available through 1979. More recently Duffy (1993) compares three popular methods to numerically invert the Laplace transform. The methods examined in Duffy are (a) the Crump inversion method, Crump (1970); (b) the Weeks method that integrates the Bromwich's integral by using Laguerre polynomials, Weeks (1966); (c) the Talbot method that deforms the Bromwich's contour so that it begins and ends in the third and second quadrant of the γ-plane, Talbot (1979). If the locations of the singularities are known, these schemes may provide accurate results at minimal computational expense. However, the user must provide a numerical value for some parameters and therefore an automatic inversion procedure is not possible. At this regard, a recent paper by Weideman (1999) seems to give more insights about the choice of the free parameters. Another simple algorithm to invert Laplace transforms is given in Den Iseger (2006). In general this algorithm outperforms the Abate–Whitt algorithm in stability and accuracy. The strength of the Den Iseger algorithm is the fact that in essence it boils down to an application of the discrete FFT algorithm. However, the Den Iseger algorithm may also perform unsatisfactorily when the function or its derivative has discontinuities. Other interesting numerical inversion algorithms can be found in Abate, Choudhury and Whitt (1996), Garbow et al. (1988a, 1988b). Finally, we mention the often quoted Gaver–Stehefest algorithm, Gaver, Jr. (1966) and Stehfest (1970), a relatively simple numerical inversion method using only values of the Laplace transform on the real axis but requiring high precision.[4]

The numerical inversion of multidimensional Laplace transforms is studied in Abate, Choudhury and Whitt (1998), Choudhury, Lucantoni and Whitt (1994), Singhal and Vlach (1975), Singhal, Vlach and Vlach (1975), Vlach and Singhal (1993), Chpt. 10, Moorthy (1995a, 1995b). Among the others, papers that discuss the instability of the numerical inversion are Bellman, Kalaba and Lockett (1966), Platzman, Ammons and Bartholdi (1988), Kwok and Barthez (1989), Craig and Thompson (1994). An useful source for the solution of ordinary differential equations is Ince (1964).

Selby (1983) and Buser (1986) have introduced the Laplace transform in finance. Useful references are Shimko (1991) and Fusai (2001), that have lots of examples on which to practice. Laplace transform has been used in finance for pricing (a) barrier options, Geman and Yor (1996), Pelsser (2000), and Sbuelz (1999, 2005), Davydov and Linetsky (2001a, 2001b); (b) interest rate derivatives, Leblanc and Scaillet (1998) and Cathcart (1998); (c) Asian options, Geman and Yor (1993), Geman and Eydeland (1995), Fu, Madan and Wang (1998), Lipton (1999), and Fusai (2004); (d) other exotic options (corridor, quantile, parisian and step options), Akahori (1995), Ballotta (2001), Ballotta and Kyprianou (2001), Chesney et al. (1995), Chesney et al. (1997), Dassios (1995), Hugonnier (1999), Linetsky (1999), Fusai (2000), Fusai and Tagliani (2001); (e) credit risk, Di Graziano and Rogers (2005); (f) options on hedge funds, Atlan, Geman and Yor (2005). A review can be found in Craddock, Heath and Platen (2000). Useful formulae related to the Laplace trans-

[4] A Matlab® implementation can be found at http://www.mathworks.com/matlabcentral/fileexchange/loadFile.do?objectId=9987&objectType=file

form of the hitting time distribution and to exponential functionals of the Brownian motion can be found in Yor (1991), Rogers (2000), Borodin and Salminen (2002), Salminen and Wallin (2005).

Finally, we mention the web page mantained by Valko,[5] a useful reference for finding the most important algorithms for the numerical inversion of the Laplace transform.

[5] http://pumpjack.tamu.edu/valko/public_html/Nil/index.html